

UNIVERSUM PRESCRIPTION: REGULARIZATION USING UNLABELED DATA

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ABSTRACT

This paper shows that simply prescribing “none of the above” labels to unlabeled data has a beneficial regularization effect to supervised learning. We call it universum prescription by the fact that the prescribed labels cannot be one of the supervised labels. In spite of its simplicity, universum prescription obtained competitive results in training deep convolutional networks for CIFAR-10, CIFAR-100, STL-10 and ImageNet datasets. A qualitative justification of these approaches using Rademacher complexity is presented. The effect of a regularization parameter – probability of sampling from unlabeled data – is also studied empirically.

1 INTRODUCTION

The idea of exploiting the wide abundance of unlabeled data to improve the accuracy of supervised learning tasks is a very natural one. In this paper, we study what is perhaps the simplest way to exploit unlabeled data in the context of deep learning. We assume that the unlabeled samples do not belong to any of the categories of the supervised task, and we force the classifier to produce a “none of the above” output for these samples. This is by no means a new idea, but we show empirically and theoretically that doing so has a beneficial regularization effect on supervised task and reduces the generalization gap, the expected difference between the test error and the training error. We study three different ways to prescribe “none of the above” outputs, dubbed uniform prescription, dustbin class, and background class and show that they improve the test error of convolutional networks trained on CIFAR-10, CIFAR-100 (Krizhevsky (2009)), STL-10 (Coates et al. (2011)), and ImageNet (Russakovsky et al. (2015)). The method is justified theoretically using Rademacher complexity (Bartlett & Mendelson (2003)).

To briefly describe the three universum prescription methods, uniform prescription forces a discrete uniform distribution across classes using cross-entropy loss. Dustbin class simply adds an extra class to the problem and prescribe all extra data to this class. Background class also adds an extra class, but it uses a constant threshold to avoid parameterization.

Our work is a direct extension to learning in the presence of universum (Weston et al. (2006)) (Chapelle et al. (2007)), originated from Vapnik (1998) and Vapnik (2006). The definition of universum is a set of unlabeled data that are known not to belong to any of the classes but in the same domain. We extended the idea of using universum from support vector machines to deep learning.

Using unlabeled data to facilitate supervised learning is sometimes called semi-supervised learning as surveyed by Chapelle et al. (2006b) and Zhu & Goldberg (2009). The most related ones are information regularization (Corduneanu & Jaakkola (2006)) and transduction learning (Chapelle et al. (2006a)) (Gammerman et al. (1998)). In these approaches, prescribing supervised labels to unlabeled data is part of the overall algorithm. They are the opposite case of universum prescription.

Representation or feature learning (reviewed by Bengio et al. (2013) and Bengio & LeCun (2007)) and transfer learning (Thrun & Pratt (1998)) are also related to our work. They include the idea of pretraining (Erhan et al. (2010) Hinton et al. (2006) Ranzato et al. (2006)), which transfers the features learnt from unlabeled data to some supervised task. Universum prescription incorporates unlabeled data as part of the supervised training process, imposing neither sparsity nor reconstruction.

The methods in this article could be thought of as a simple form of multi-task learning (Baxter (2000)) (Caruana (1993)), where an auxiliary task is to control overfitting under the universum assumption (see section 2). It can also be thought of as using hints (Abu-Mostafa (1990)) (Suddarth & Holden (1991)) for training where the hint is functional regularity from unlabeled data.

Universum prescription is also related to the idea of distillation or “dark knowledge” (Bucilu et al. (2006)) (Hinton et al. (2015)). The idea is to prescribe “soft” targets from an ensemble of models to a single model, and improvement on classification accuracy is observed. Uniform prescription prescribes “soft” targets to unlabeled data as well, except that the targets are agnostic to the classification problem.

Regularization – techniques for the control of overfitting or generalization gap – has been studied extensively. Most of the practical approaches implement a secondary optimization objective, such as L_1 or L_2 norm. Some other methods such as dropout (Srivastava et al. (2014)) and dropconnect (Wan et al. (2013)) cheaply simulate model averaging to control the model variance.

As part of the general statistical learning theory (Vapnik (1995), Vapnik (1998)), the justification for regularization is well-developed. There are many formulations, such as probably approximately correct (PAC) learning (Valiant (1984)), the trade-off between bias and variance (Geman et al. (1992)), and the prescription of Bayesian a priori (Mozier & Smolensky (1989)). We qualitatively justify the methods using Radamacher complexity (Bartlett & Mendelson (2003)), similar to Wan et al. (2013).

2 UNIVERSUM PRESCRIPTION

In this section we attempt to formalize the the trick of prescribing “none of the above” labels. We call it universum prescription because these labels could not belong to any supervised class. Consider the problem of exclusive k -way classification. In inference we can find the most probable class $y \in \{1, 2, \dots, k\}$ given input x . In learning we hope to find a hypothesis function $h \in \mathcal{H}$ mapping to \mathbb{R}^k so that the label is determined by $y = \operatorname{argmin}_i h_i(x)$. The following assumptions are made.

1. (Loss assumption) The loss used as the optimization objective is negative log-likelihood:

$$L(h, x, y) = h_y(x) + \log \left[\sum_{i=1}^k \exp(-h_i(x)) \right]. \quad (1)$$

2. (Universum assumption) The proportion of samples belonging to one of the k classes in the unlabeled data is negligible.

The loss assumption assumes that the probability of class y given an input x can be thought of as

$$\Pr[Y = y|x, h] = \frac{\exp(-h_y(x))}{\sum_{i=1}^k \exp(-h_i(x))}, \quad (2)$$

where $(X, Y) \sim \mathbf{D}$ and \mathbf{D} is the distribution where labeled data are sampled. We use lowercase letters for values, uppercase letters for random variables and bold uppercase letters for distribution. The loss assumption is simply a necessary detail rather than a limitation, in the sense that one can change the type of loss and use the same principles to derive different universum learning techniques.

The universum assumption implicates that labeled classes are a negligible subset. In many practical cases we only care about a small number of classes, either by problem design or due to high cost in the labeling process. At the same time, a very large amount of unlabeled data is easily obtained. Put in mathematics, assuming we draw unlabeled data from distribution \mathbf{U} , the assumption states that

$$\Pr_{(X, Y) \sim \mathbf{U}} [X, Y \in \{1, 2, \dots, k\}] \approx 0. \quad (3)$$

There are several reasons for equation 3 being defined by $\Pr[X, Y]$ instead of $\Pr[Y|X]$. By $\Pr[X, Y] = \Pr[Y|X] \Pr[X]$, we know that $\Pr[X, Y]$ is ignorable if $\Pr[Y|X]$ is ignorable. However, it should also be possible for $\Pr[X, Y]$ to be ignorable because $\Pr[X]$ is ignorable, which describes the case that an ignorable portion of supervised data could be in the joint set. When applied to the loss function set, Rademacher complexity can be thought of as an expectation over the joint distribution of (X, Y) . Equation 3 is therefore a definition consistent with the theory.

The universum assumption is opposite to the assumptions of information regularization (Corduneanu & Jaakkola (2006)) and transduction learning (Chapelle et al. (2006a)) (Gammerman et al. (1998)). All the methods discussed below prescribe agnostic targets to the unlabeled data. During learning, we randomly present an unlabeled sample to the optimization procedure with probability p .

2.1 UNIFORM PRESCRIPTION

It is known that negative log-likelihood is simply a reduced form of cross-entropy

$$L(h, x, y) = - \sum_{i=1}^k Q[Y = i|x] \log \Pr[Y = i|x, h] \quad (4)$$

in which the target probability $Q[Y = y|x] = 1$ and $Q[Y = i|x] = 0$ for $i \neq y$. Under the universum assumption, if we are presented with an unlabeled sample x , we would hope to prescribe some Q so that every class has some equally minimal probability. Q also has to satisfy $\sum_{i=1}^k Q[Y = i|x] = 1$ by the probability axioms. The only possible choice for Q is then $Q[Y|x] = 1/k$. The learning algorithm then uses the cross-entropy loss instead of negative log-likelihood.

It is worth noting that uniform output has the maximum entropy among all possible choices. In the case that the hypothesis h is parameterized as a deep neural network, uniform output is achieved when these parameters are constantly 0. Therefore, uniform prescription may have the effect of reducing the magnitude of parameters, similar to norm-based regularization.

2.2 DUSTBIN CLASS

Another way of prescribing agnostic target is to append a “dustbin” class to the supervised task. This requires some changes to the hypothesis function h such that it outputs $k + 1$ targets. For deep learning models one can simply extend the last parameterized layer. All unlabeled data are prescribed to this extra “dustbin” class. The learning algorithm remains unchanged.

The effect of dustbin class is clearly seen in the loss function of an unlabeled sample $(x, k + 1)$

$$L(h, x, k + 1) = h_{k+1}(x) + \log \left[\sum_{i=1}^{k+1} \exp(-h_i(x)) \right]. \quad (5)$$

The second term is a “soft” maximum for all dimensions of $-h$. When an unlabeled sample is present, the algorithm attempts to introduce smoothness by minimizing probability spikes.

2.3 BACKGROUND CLASS

We could further simplify dustbin class by removing parameters for class $k + 1$. For some given threshold constant τ , we could change the probability of a labeled sample to

$$\Pr[Y = y|x, h] = \frac{\exp(-h_y(x))}{\exp(-\tau) + \sum_{i=1}^k \exp(-h_i(x))}, \quad (6)$$

and an unlabeled sample

$$\Pr[Y = k + 1|x, h] = \frac{\exp(-\tau)}{\exp(-\tau) + \sum_{i=1}^k \exp(-h_i(x))}. \quad (7)$$

This will result in changes to the loss function of a labeled sample (x, y) as

$$L(h, x, y) = h_y(x) + \log \left[\exp(-\tau) + \sum_{i=1}^k \exp(-h_i(x)) \right], \quad (8)$$

and an unlabeled sample

$$L(h, x, k + 1) = \tau + \log \left[\exp(-\tau) + \sum_{i=1}^k \exp(-h_i(x)) \right]. \quad (9)$$

We call this method background class and τ background constant. Similar to dustbin class, the algorithm attempts to minimize the spikes of outputs, but limited to a certain extent by the inclusion of $\exp(-\tau)$ in the partition function. In our experiments τ is always set to 0.

3 THEORETICAL JUSTIFICATION

In this part, we derive a qualitative justification for universum prescription using probably approximately correct (PAC) learning (Valiant (1984)). By a “qualitative” theory, we are comparing with numerical bounds such as growth function (Massart (2000), Vapnik (1998)), Vapnik-Chervonenkis dimension (Vapnik & Chervonenkis (1971)), covering numbers (Dudley (1967)) and others. Our theory is based on Rademacher complexity (Bartlett & Mendelson (2003)), similar to Wan et al. (2013) where both dropout (Srivastava et al. (2014)) and dropconnect (Wan et al. (2013)) are justified. Rademacher complexity is usually a lower-bound of other numerical complexity measurement. Previous results on unlabeled data (Oneto et al. (2011)) (Oneto et al. (2015)) assume that labeled and unlabeled data follow the same distribution, which is impossible under the universum assumption.

Definition 1 (Empirical Rademacher complexity). *Let \mathcal{F} be a family of functions mapping from \mathcal{U} to \mathbb{R} , and $S = (x_1, x_2, \dots, x_m)$ a fixed sample of size m with elements in \mathcal{X} . Then, the empirical Rademacher complexity of \mathcal{F} with respect to the sample S is defined as:*

$$\hat{\mathfrak{R}}_S(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\eta}} \left[\sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^m \eta_i f(x_i) \right] \quad (10)$$

where $\boldsymbol{\eta} = (\eta_1, \dots, \eta_m)^T$, with η_i ’s independent random variables taking values from a discrete uniform distribution on $\{-1, 1\}$.

Definition 2 (Rademacher complexity). *Let \mathbf{D} denote the distribution from which the samples were drawn. For any integer $m \geq 1$, the Rademacher complexity of \mathcal{F} is the expectation of the empirical Rademacher complexity over all samples of size m drawn according to \mathbf{D} :*

$$\mathfrak{R}_m(\mathcal{F}, \mathbf{D}) = \mathbb{E}_{S \sim \mathbf{D}^m} [\hat{\mathfrak{R}}_S(\mathcal{F})] \quad (11)$$

It could be argued that the distribution for $\boldsymbol{\eta}$ is arbitrary. There are other possibilities such as Gaussian complexity (Bartlett & Mendelson (2003)), but they can all be generalized to “stochastic complexity” and result in the same conclusions (Zhang (2013)). In the case that f has multiple outputs, one can simply add the complexity measurements for each outputs together and the theory still holds.

Two qualitative properties of Rademacher complexity is worth noting here. First of all, Rademacher complexity is always non-negative by the convexity of supremum

$$\hat{\mathfrak{R}}_S(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\eta}} \left[\sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^m \eta_i f(x_i) \right] \geq \sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^m \mathbb{E}_{\eta_i} [\eta_i] f(x_i) = 0. \quad (12)$$

Secondly, if for a fixed input all functions in \mathcal{F} output the same value, then it’s Rademacher complexity is 0. Assume for any $f \in \mathcal{F}$ we have $f(x) = f_0(x)$, then

$$\hat{\mathfrak{R}}_S(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\eta}} \left[\sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^m \eta_i f(x_i) \right] = \mathbb{E}_{\boldsymbol{\eta}} \left[\sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^m \eta_i f_0(x) \right] = \frac{1}{m} \sum_{i=1}^m \mathbb{E}_{\eta_i} [\eta_i] f_0(x) = 0. \quad (13)$$

Therefore, one way to qualitatively minimize Rademacher complexity is to regularize functions in \mathcal{F} such that all functions tend to have the same output for a given input. Universum prescription precisely does that – the prescribed outputs for unlabeled data are all constantly the same.

The principal PAC-learning result from literature is an approximation bound for function spaces that has finite bounds for outputs. We use the formulation by Zhang (2013), but anterior results are in Bartlett et al. (2002), Bartlett & Mendelson (2003), Koltchinskii (2001) and Koltchinskii & Panchenko (2000). We refer the reader to these publications for proof.

Table 1: The 21-layer network

LAYERS	DESCRIPTION
1-3	Conv 256x3x3
4	Pool 2x2
5-8	Conv 512x3x3
9	Pool 2x2
10-13	Conv 1024x3x3
14	Pool 2x2
15-18	Conv 1024x3x3
19	Pool 2x2
20-23	Conv 2048x3x3
24	Pool 2x2
25-26	Full 2048

Theorem 1 (Approximation bound with finite bound on output). *For a well-defined objective $\mathcal{E}(h, x, y)$ over hypothesis class \mathcal{H} , input set \mathcal{X} and output set \mathcal{Y} , if it has an upper bound $M > 0$, then with probability at least $1 - \delta$, the following holds for all hypothesis $h \in \mathcal{H}$:*

$$\mathbb{E}_{(x,y) \sim \mathbf{D}}[\mathcal{E}(h, x, y)] \leq \frac{1}{m} \sum_{(x,y) \in S} \mathcal{E}(h, x, y) + 2\mathfrak{R}_m(\mathcal{F}, \mathbf{D}) + M \sqrt{\frac{\log \frac{2}{\delta}}{2m}}, \quad (14)$$

where the function family \mathcal{F} is defined as

$$\mathcal{F} = \{\mathcal{E}(h, x, y) | h \in \mathcal{H}\}, \quad (15)$$

\mathbf{D} is a distribution on the samples (x, y) , and S is a set of samples of size m drawn indentially and independently from \mathbf{D} .

In the theorem above, the objective functional $\mathcal{E}(h, x, y)$ should be lower-bounded by 0, and it corresponds to a negatively correlated compatibility measurement between a hypothesis h and a sample (x, y) . It is similar to the definition of energy used by energy-based learning in LeCun et al. (2006). It could be the error function $\mathcal{E}(h, x, y) = 1 - \mathbf{1}\{y = \arg\min_i(h_i(x))\}$, the exponential function $\mathcal{E}(h, x, y) = \exp(h_y(x))$, the negative probability function $\mathcal{E}(h, x, y) = 1 - \Pr[Y = y|x, h]$, or simply the loss $\mathcal{E}(h, x, y) = L(h, x, y)$, which is the negative log-probability.

For some choices of \mathcal{E} we have a bound $\mathcal{E}(h, x, y) \leq M$ by design, whereas for some others it is more intricate to believe M exists. If the learning algorithm is an iterative optimization procedure such as gradient descent, at each step one could believe that a limit M exists relatively to the current hypothesis h_0 (Zhang (2013)). This is because of the dynamics of iterative optimization – the algorithm can only explore some sublevel hypothesis set \mathcal{H}_0 in later steps.

The meaning of the theorem is two-folded. When applying the theorem to the joint problem of training using both labeled and unlabeled data, the third term on the right hand of inequality 14 is reduced by the augmentation of the extra data. The joint problem can be written as $(x, y) \sim (1 - p)\mathbf{D} + p\mathbf{U}$. The value of the term $\mathfrak{R}_m(\mathcal{F}, (1 - p)\mathbf{D} + p\mathbf{U})$ is reduced when we prescribe constant outputs, due to the qualitative properties of Rademacher complexity discussed before.

The second fold is that when the theorem applies to the supervised distribution \mathbf{D} , we would hope that $\mathfrak{R}_n(\mathcal{F}, \mathbf{D})$ can be bounded by $\mathfrak{R}_m(\mathcal{F}, (1 - p)\mathbf{D} + p\mathbf{U})$, where n is the number of supervised samples randomly chosen by the joint problem. Note that the number n follows a binomial distribution with mean $(1 - p)m$. Such a bound can be achieved in a probable and approximate sense.

Theorem 2 (Rademacher complexity bound on distribution mixture). *Assume we have a joint problem where $p \leq 0.5$ and there are m random training samples from the joint distribution $(1 - p)\mathbf{D} + p\mathbf{U}$. With probability at least $1 - \delta$, the following holds*

$$\mathfrak{R}_n(\mathcal{F}, \mathbf{D}) \leq \frac{2 - p}{(1 - p) \left(1 - p - \sqrt{\frac{\log(1/\delta)}{2m}}\right)} \mathfrak{R}_m(\mathcal{F}, (1 - p)\mathbf{D} + p\mathbf{U}), \quad (16)$$

where n is a random number indicating the number of supervised samples in the total joint samples, and m is large enough such that

$$1 - p - \sqrt{\frac{\log(1/\delta)}{2m}} > 0. \quad (17)$$

We present the proof of theorem 2 in the appendix. The theorem tells us that the Rademacher complexity of the supervised portion of the joint problem can be bound by the total joint Rademacher complexity. The universum prescription algorithm attempts to make the Rademacher complexity

Table 2: The 17-layer network

LAYERS	DESCRIPTION
1-3	Conv 128x3x3
4	Pool 2x2
5-7	Conv 256x3x3
8	Pool 2x2
9-11	Conv 512x3x3
12	Pool 2x2
13-15	Conv 1024x3x3
16	Pool 2x2
17-19	Conv 2048x3x3
20	Pool 2x2
21-22	Full 4096

of the joint problem small. Therefore, for different sample sizes of labeled and unlabeled data, universum prescription may bring improvement for generalization.

Theorem 2 also has implications for asymptotic bounds. It is long known to the machine learning community that the following holds (for example, combining Massart (2000) and the idea of growth function Vapnik (1998))

$$\mathfrak{R}_n(\mathcal{F}, \mathbf{D}) \leq \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), \quad (18)$$

if the function family \mathcal{F} is upper- and lower-bounded. This is satisfied by our assumption that $0 \leq \mathcal{E}(h, x, y) \leq M$. By the form of inequality 16 and the fact that $\mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{U}) \leq \mathcal{O}(1/\sqrt{m})$, it is easy to know that instead of inequality 18, we can achieve the asymptotic bound

$$\mathfrak{R}_n(\mathcal{F}, \mathbf{D}) \leq \mathcal{O}\left(\frac{1}{\sqrt{m}}\right). \quad (19)$$

Note that n is a random number indicating the portion of supervised samples in the total of m joint samples. This means that by solving the joint problem, we essentially provide the supervised problem a superior generalization guarantee with an asymptotic relation with the total sample number m , including both supervised and unsupervised samples.

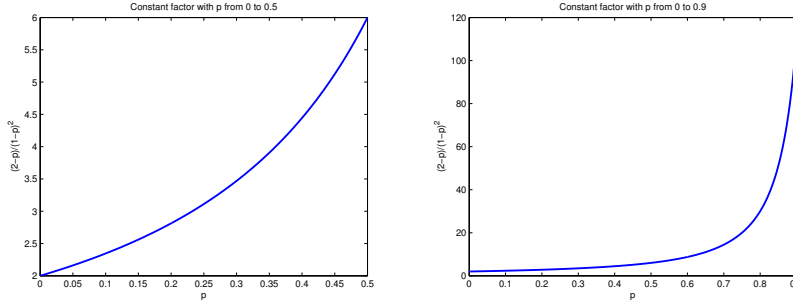


Figure 1: The asymptotic constant factor in inequality 16.

However, theorem 2 has a requirement that $p \leq 0.5$, otherwise all the bounds discussed above are not achievable. Figure 1 shows the value of $(2-p)/(1-p)^2$ for different choice of p , which is the asymptotic constant factor in inequality 16 when m is large. The factor is monotonically increasing with respect to p , within the range of $[2, 6]$ when $p \leq 0.5$. These properties indicate that we need to keep p small to prevent the bound from collapsing. Experiments in section 5 show that there is an improvement with testing error if p is small (up to around 0.3 to 0.4), but both training and testing errors became worse with p larger than a certain value.

It should also be noted that the asymptotic constant factor does not seem to be tight at $p = 0$, in which case it should obviously be 1. This suggests that a improvement for the constant factor might be possible, depending on whether the case at $p = 0$ can be analyzed with contiguity.

4 EXPERIMENTS ON IMAGE CLASSIFICATION

In this section we test the methods on some image classification tasks. Three series of datasets – CIFAR-10/100 (Krizhevsky (2009)), STL-10 (Coates et al. (2011)) and ImageNet (Russakovsky et al. (2015)) – are chosen due to the availability of unlabeled data. For CIFAR-10/100 and STL-10 datasets, we used a 21-layer convolutional network (ConvNet) (LeCun et al. (1989), LeCun et al. (1998)), in which the inputs are 32-by-32 images and all convolutional layers are 3-by-3 and fully padded. For ImageNet, the model is a 17-layer ConvNet with 64-by-64 images as inputs. These models are inspired by Simonyan & Zisserman (2014), in which all pooling layers are max-pooling, and ReLUs (Nair & Hinton (2010)) are used as the non-linearity after all convolutional and linear layers. Two dropout (Srivastava et al. (2014)) layers of probability 0.5 are inserted before the final two linear layers.

Table 3: Result for universum prescription. The numbers are percentages. The three numbers in each tabular indicate training error, testing error and generalization gap. Bold numbers are the best ones for each case. CIFAR-100 F. and CIFAR-100 C. stand for fine-grained and coarse classification problems of CIFAR-100. STL-10 Tiny stands for using 80 million images as the unlabeled dataset. ImageNet-1 and ImageNet-5 are errors for top-1 and top-5 evaluation for ImageNet dataset.

DATASET	BASELINE			UNIFORM			DUSTBIN			BACKGROUND		
	Train	Test	Gap	Train	Test	Gap	Train	Test	Gap	Train	Test	Gap
CIFAR-10	0.00	7.02	7.02	0.72	7.59	6.87	0.07	6.66	6.59	1.35	8.38	7.03
CIFAR-100 F.	0.09	37.58	37.49	4.91	36.23	31.32	2.52	32.84	30.32	8.56	40.57	42.01
CIFAR-100 C.	0.04	22.74	22.70	0.67	23.42	22.45	0.40	20.45	20.05	3.73	24.97	21.24
STL-10	0.00	31.16	31.16	2.02	36.54	34.52	3.03	36.58	33.55	14.89	38.95	24.06
STL-10 Tiny	0.00	31.16	31.16	0.62	30.15	29.47	0.00	27.96	27.96	0.11	30.38	30.27
ImageNet-1	10.19	34.39	24.20	13.84	34.61	20.77	13.80	33.67	19.87	13.43	34.69	21.26
ImageNet-5	1.62	13.68	12.06	3.02	13.70	10.68	2.83	13.35	10.52	2.74	13.84	11.10

The algorithm used is stochastic gradient descent with momentum (Polyak (1964), Sutskever et al. (2013)) 0.9 and a minibatch size of 32. The initial learning rate is 0.005 which is halved every 60,000 minibatch steps for CIFAR-10/100 and every 600,000 minibatch steps for ImageNet. The training stops at 400,000 minibatch steps for CIFAR-10/100 and STL10, and 2,500,000 minibatch steps for ImageNet. Table 1 and 2 summarize the configurations. For a convolutional layer the number of output feature maps is shown, and for a linear layer the number of hidden units. The weights are initialized in the same way as He et al. (2015).

The initial motivation for choosing networks like such is to make sure they will have enough capacity for overfitting so that the effect of regularization is clearly shown. However, in practice such large networks already has a very good baseline even without universum prescription. This is probably due to the data augmentation steps below, which are used in all our experiments.

1. (Horizontal flip.) Flip the image horizontally with probability 0.5.
2. (Scale.) Randomly scale the image between 1/1.2 and 1.2 times of its height and width.
3. (Crop.) Randomly crop a 32-by-32 (or 64-by-64 for ImageNet) region in the scaled image.

4.1 CIFAR-10 AND CIFAR-100

The samples of CIFAR-10 and CIFAR-100 datasets (Krizhevsky (2009)) are from the 80 million tiny images dataset (Torralba et al. (2008)). Each dataset contains 60,000 samples, constituting a very small portion of 80 million. This is an ideal case for our methods, in which we can use the entire 80 million images as the unlabeled data. The CIFAR-10 dataset has 10 classes, and CIFAR-100 has 20 (coarse) or 100 (fine-grained) classes. Table 3 contains the results. The generalization gap is approximated by the difference between testing and training errors. All of the universum prescription models use unlabeled data with probability $p = 0.2$.

Table 4: Comparison of single-model CIFAR-10 and CIFAR-100 results, in second and third columns. The fourth column indicates whether data augmentation is used for CIFAR-10. The numbers are percentages.

METHOD	10	100	AUG.
Graham (2014)	6.28	24.30	YES
Dustbin class (ours)	6.66	32.84	YES
Lee et al. (2015)	7.97	34.57	YES
Lin et al. (2013)	8.81	35.68	YES
Goodfellow et al. (2013)	9.38	38.57	YES
Wan et al. (2013)	11.10	N/A	NO
Zeiler & Fergus (2013)	15.13	42.51	NO

We compared other single-model results on CIFAR-10 and CIFAR-100 (fine-grained case) in table 4. It shows that our network is competitive to the state of the art.

4.2 STL-10

The STL-10 dataset (Coates et al. (2011)) has size 96-by-96 for its images. We downsampled them to 32-by-32 so as to use the same model. The dataset contains a very small number of training samples – 5000 in total. The accompanying unlabeled dataset is larger with 100,000 samples. There is no guarantee that these extra samples are outside of the supervised training classes. Universum prescription failed in this case.

To verify that the extra data is the problem, we also performed an experiment using the 80 million tiny images as the unlabeled dataset, as shown in table 3. Due to long training times of our models, we did not perform 10-fold training as in the original paper by Coates et al. (2011), therefore our result is not comparable to those in the literature. We present them only to show the effectiveness of universum prescription influenced by the universum assumption on the unlabeled data.

One interesting observation is that the results on STL-10 became better with the use of 80 million tiny images instead of the original extra data. It indicates that dataset size and whether universum assumption is satisfied are affecting factors for the effectiveness of universum prescription.

4.3 IMAGENET

The ImageNet dataset (Russakovsky et al. (2015)) for classification task has in total 1,281,167 training images and 50,000 validation images. The testing errors reported are evaluated on this validation dataset. During training, we resize all images to minimum weight or height 64, and then feed a random 64-by-64 crop to our network after applying the previous data augmentation steps. During testing, we apply the same test-time augmentation technique as in Szegedy et al. (2015) with size variants {64, 72, 80, 88}, where each image is viewed in 144 different crops.

The extra data comes from the large ImageNet 2011 release¹, in which we remove all images that belong to a class or a parent of a class of the classification task. This is enabled by the super-subordinate (is-a) relation information provided with the WordNet distribution Miller (1995), since all ImageNet classes are nouns of WordNet. The resulting extra dataset contains 12,772,763 images. The results of ImageNet dataset are reported in table 3 for both top-1 and top-5 errors.

In all experiments dustbin class provides surprisingly good results. There are many factors that could result in this, such as the choice of p and the choice of threshold value for back-ground class. It is computationally infeasible for a complete analysis in the scope of this paper, but future research is certainly warranted.

5 EFFECT OF THE REGULARIZATION PARAMETER

One natural question to ask of our models is how would change of the probability p of sampling from unlabeled data affect the results. In this section we show the experiments. To prevent an exhaustive search on the regularization parameter from overfitting our models on the testing data, we use a different model for this section. It is described in table 5, which has 9 parameterized layers in total. The design is inspired by Sermanet et al. (2013). For each choice of p we conducted 6 experiments combining universum prescription models and dropout. The dropout layers are two ones added in between the fully-connected layers with dropout probability 0.5. Figure 2 shows the results.

Table 5: ConvNet for section 5

LAYERS	DESCRIPTION
1	Conv 1024x5x5
2	Pool 2x2
3	Conv 1024x5x5
4-7	Conv 1024x3x3
8	Pool 2x2
9-11	Full 2048

From figure 2 we can conclude that increasing p will decrease generalization gap. However, we cannot make p too large since after a certain point the training collapses and both training and testing errors become worse. This confirms with the assumptions and conclusions from theorem 2.

Comparing between CIFAR-10/100 and STL-10, one conclusion is that the model variance is affected by the combined size of labeled and unlabeled datasets. The variance on training and testing

¹<http://www.image-net.org/releases>

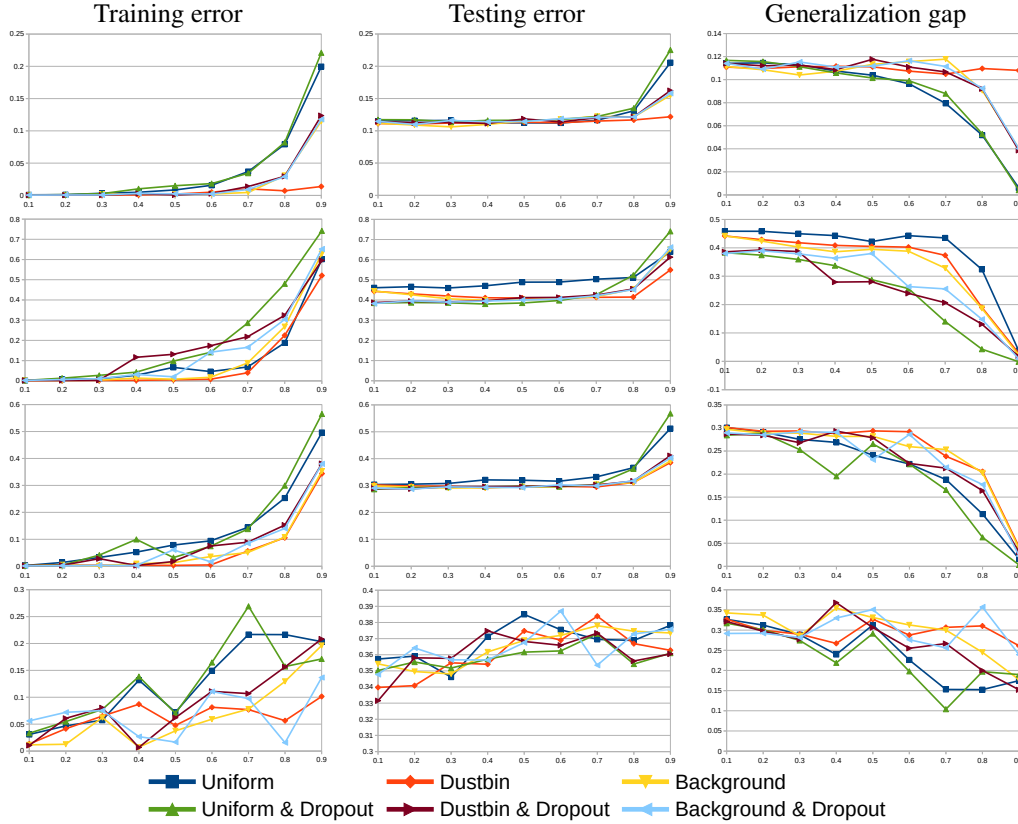


Figure 2: Experiments on regularization parameter. The four rows are CIFAR-10, CIFAR-100 fine-grained, CIFAR-100 coarse and STL-10 respectively.

errors are extremely small on CIFAR-10/100 datasets because the extra data we used is almost unlimited (in total 80 million), but on STL-10 the variance seems to be large with much smaller combined size of training and extra datasets. This suggests that using universum prescription with a large abundance of extra data could improve the stability of supervised learning algorithms.

Finally, the comparison between using and not using dropout does not show a difference that is in anyway statistically significant. This suggests that the regularization effect of universum prescription is comparable to that of dropout. There is therefore evident effectiveness of a data-driven regularization method compared to data-agnostic regularization methods.

6 CONCLUSION AND OUTLOOK

This article shows that universum prescription can be used to regularize a multi-class classification problem using extra unlabeled data. Two assumptions are made, in which one is that loss used is negative log-likelihood and the other is negligible probability of a supervised sample existing in the unlabeled data. The loss assumption is a necessary detail rather than a limitation. The three universum prescription methods are uniform prescription, dustbin class and background class. Uniform prescription forces a discrete uniform distribution across classes using cross-entropy loss. Dustbin class simply adds an extra class to the problem and prescribe all extra data to this class. Background class also adds an extra class, but it uses a constant threshold to avoid parameterization.

We further provided a theoretical justification. Theorem 2 suggests that asymptotically the generalization ability of the supervised problem could be bounded by the joint problem, which has more samples due to the addition of unsupervised (unlabeled) data. Experiments are done using CIFAR-10, CIFAR-100 and STL-10 datasets. The effect of the regularization parameter is also studied empirically. These experiments show that all three universum prescription methods provide certain

improvement over the generalization gap, whereas dustbin class constantly performs the best. Further conclusions include that additional unlabeled data can improve the variance of models during training, and that the results are comparable to data-agnostic regularization using dropout.

In the future, we hope to apply these methods to a broader range of problems.

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APPENDIX: PROOF OF THEOREM 2

As an outline of our proof, we first establish a relation between $\mathfrak{R}_m(\mathcal{F}, \mathbf{D})$ and $\mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{U})$, and then another relation between $\mathfrak{R}_n(\mathcal{F}, \mathbf{D})$ and $\mathfrak{R}_m(\mathcal{F}, \mathbf{D})$. The first part requires the following lemmas.

Lemma 1 (Separation of dataset on empirical Rademacher complexity). *Let S be a dataset of size m . If S_1 and S_2 are two non-overlap subset of S such that $|S_1| = m - i$, $|S_2| = i$ and $S_1 \cup S_2 = S$, then the following two inequalities hold*

$$\hat{\mathfrak{R}}_S(\mathcal{F}) \leq \frac{m-i}{m} \hat{\mathfrak{R}}_{S_1}(\mathcal{F}) + \frac{i}{m} \hat{\mathfrak{R}}_{S_2}(\mathcal{F}). \quad (20)$$

Proof. Let $(x_j, y_j) \in S_1$ for $j = 1, 2, \dots, m-i$ and $(x_j, y_j) \in S_2$ for $j = m-i+1, m-i+2, \dots, m$. Denote \mathbf{N} as the discrete uniform distribution on $\{1, -1\}$. We can derive by the convexity of supremum and symmetry of \mathbf{N}

$$\begin{aligned} \hat{\mathfrak{R}}_S(\mathcal{F}) &= \mathbb{E}_{\boldsymbol{\eta} \sim \mathbf{N}^m} \left[\sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{j=1}^m \eta_j f(x_j) \right] \\ &= \frac{2}{m} \mathbb{E}_{\boldsymbol{\eta} \sim \mathbf{N}^m} \left[\sup_{f \in \mathcal{F}} \left(\frac{1}{2} \sum_{j=1}^{m-i} \eta_j f(x_j) + \frac{1}{2} \sum_{j=m-i+1}^m \eta_j f(x_j) \right) \right] \\ &\leq \frac{2}{m} \mathbb{E}_{\boldsymbol{\eta} \sim \mathbf{N}^m} \left[\frac{1}{2} \sup_{f \in \mathcal{F}} \left(\sum_{j=1}^{m-i} \eta_j f(x_j) \right) + \frac{1}{2} \sup_{f \in \mathcal{F}} \left(\sum_{j=m-i+1}^m \eta_j f(x_j) \right) \right] \\ &= \frac{m-i}{m} \mathbb{E}_{\boldsymbol{\eta} \sim \mathbf{N}^{m-i}} \left[\sup_{f \in \mathcal{F}} \frac{1}{m-i} \sum_{j=1}^{m-i} \eta_j f(x_j) \right] + \frac{i}{m} \mathbb{E}_{\boldsymbol{\eta} \sim \mathbf{N}^i} \left[\sup_{f \in \mathcal{F}} \frac{1}{i} \sum_{j=m-i+1}^m \eta_j f(x_j) \right] \\ &= \frac{m-i}{m} \hat{\mathfrak{R}}_{S_1}(\mathcal{F}) + \frac{i}{m} \hat{\mathfrak{R}}_{S_2}(\mathcal{F}). \end{aligned}$$

□

Lemma 2 (Sample size inequality for Rademacher complexity). *Assume $0 \leq n \leq m$. If $|S_n| = n$, $|S_m| = m$ and $S_m = S_n \cup \{x_{n+1}, x_{n+2}, \dots, x_m\}$, then*

$$n \hat{\mathfrak{R}}_{S_n}(\mathcal{F}) \leq m \hat{\mathfrak{R}}_{S_m}(\mathcal{F}), \quad (21)$$

and

$$n \mathfrak{R}_n(\mathcal{F}, \mathbf{D}) \leq m \mathfrak{R}_m(\mathcal{F}, \mathbf{D}). \quad (22)$$

Proof. First of all, it is obvious that inequality 22 can be established using mathematical induction if we have $m \mathfrak{R}_m(\mathcal{F}, \mathbf{D}) \leq (m+1) \mathfrak{R}_{m+1}(\mathcal{F}, \mathbf{D})$ for all $m \geq 0$. To prove this, we first establish that if $S_m = \{x_1, x_2, \dots, x_m\}$ and $S_{m+1} = \{x_1, x_2, \dots, x_m, x_{m+1}\}$ (i.e., $S_{m+1} = S_m \cup \{x_{m+1}\}$), then $m \hat{\mathfrak{R}}_{S_m}(\mathcal{F}) \leq (m+1) \hat{\mathfrak{R}}_{S_{m+1}}(\mathcal{F})$, which can also establish inequality 21.

For any $\boldsymbol{\eta}_m = \{\eta_1, \eta_2, \dots, \eta_m\}$ and $\boldsymbol{\eta}_{m+1} = \{\eta_1, \eta_2, \dots, \eta_m, \eta_{m+1}\}$, that is, $\boldsymbol{\eta}_{m+1} = \boldsymbol{\eta}_m \cup \{\eta_{m+1}\}$, let $f_0 = \arg\max_{f \in \mathcal{F}} \sum_{i=1}^m \eta_i f(x_i)$. By definition of supremum, we have

$$\sup_{f \in \mathcal{F}} \sum_{i=1}^{m+1} \eta_i f(x_i) \geq \sum_{i=1}^{m+1} \eta_i f_0(x_i) = \sum_{i=1}^m \eta_i f_0(x_i) + \eta_{m+1} f_0(x_{m+1}) = \sup_{f \in \mathcal{F}} \sum_{i=1}^m \eta_i f(x_i) + \eta_{m+1} f_0(x_{m+1}).$$

Taking expectation over $\boldsymbol{\eta}_{m+1}$, by the symmetry of distribution \mathbf{N} , we obtain

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\eta}_{m+1} \sim \mathbf{N}^{m+1}} \left[\sup_{f \in \mathcal{F}} \sum_{i=1}^{m+1} \eta_i f(x_i) \right] &\geq \mathbb{E}_{\boldsymbol{\eta}_{m+1} \sim \mathbf{N}^m} \left[\sup_{f \in \mathcal{F}} \sum_{i=1}^m \eta_i f(x_i) + \eta_{m+1} f_0(x_{m+1}) \right] \\ &= \mathbb{E}_{\boldsymbol{\eta}_m \sim \mathbf{N}^m} \left[\sup_{f \in \mathcal{F}} \sum_{i=1}^m \eta_i f(x_i) \right] + \mathbb{E}_{\eta_{m+1} \sim \mathbf{N}} [\eta_{m+1}] f_0(x_{m+1}) \\ &= \mathbb{E}_{\boldsymbol{\eta}_m \sim \mathbf{N}^m} \left[\sup_{f \in \mathcal{F}} \sum_{i=1}^m \eta_i f(x_i) \right]. \end{aligned}$$

By the definition of $\hat{\mathfrak{R}}_{S_m}(\mathcal{F})$, the inequality above implies $m \hat{\mathfrak{R}}_{S_m}(\mathcal{F}) \leq (m+1) \hat{\mathfrak{R}}_{S_{m+1}}(\mathcal{F})$. Then, by taking expectation over S_{m+1} we can obtain

$$(m+1) \mathfrak{R}_{m+1}(\mathcal{F}, \mathbf{D}) = \mathbb{E}_{S_{m+1} \sim \mathbf{D}^{m+1}} [(m+1) \hat{\mathfrak{R}}_{S_{m+1}}] \geq \mathbb{E}_{S_m \sim \mathbf{D}^m} [m \hat{\mathfrak{R}}_{S_m}] = m \mathfrak{R}_m(\mathcal{F}, \mathbf{D}).$$

The lemma can therefore be easily established by mathematical induction. □

Using the lemmas above, the relation between $\mathfrak{R}_m(\mathcal{F}, \mathbf{D})$ and $\mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{U})$ can be established as the following theorem, by assuming $p \leq 0.5$.

Theorem 3 (Relation of Rademacher complexities in distribution mixture). *If $p \leq 0.5$, then*

$$\mathfrak{R}_m(\mathcal{F}, \mathbf{D}) \leq \frac{2-p}{1-p} \mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{U}). \quad (23)$$

Proof. For any function space \mathcal{F} and distribution \mathbf{D} , denote $\mathfrak{R}_0(\mathcal{F}, \mathbf{D}) = 0$ and $\hat{\mathfrak{R}}_\emptyset(\mathcal{F}) = 0$. By definition of Rademacher complexity and lemma 1, we get

$$\begin{aligned} \mathfrak{R}_m(\mathcal{F}, \mathbf{D}) &= \mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{D}) = \mathbb{E}_{S \sim ((1-p)\mathbf{D} + p\mathbf{D})^m} [\hat{\mathfrak{R}}_S(\mathcal{F})] \\ &= \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \mathbb{E}_{S_1 \sim \mathbf{D}^i} \left[\mathbb{E}_{S_2 \sim \mathbf{D}^{m-i}} [\hat{\mathfrak{R}}_{S_1 \cup S_2}(\mathcal{F})] \right] \\ &\leq \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \mathbb{E}_{S_1 \sim \mathbf{D}^i} \left[\mathbb{E}_{S_2 \sim \mathbf{D}^{m-i}} \left[\frac{i}{m} \hat{\mathfrak{R}}_{S_1}(\mathcal{F}) + \frac{m-i}{m} \hat{\mathfrak{R}}_{S_2}(\mathcal{F}) \right] \right] \\ &= \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \left(\mathbb{E}_{S_1 \sim \mathbf{D}^i} \left[\frac{i}{m} \hat{\mathfrak{R}}_{S_1}(\mathcal{F}) \right] + \mathbb{E}_{S_2 \sim \mathbf{D}^{m-i}} \left[\frac{m-i}{m} \hat{\mathfrak{R}}_{S_2}(\mathcal{F}) \right] \right) \\ &= \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \left[\frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) + \frac{m-i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \right] \\ &= \left[\sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \right] + \left[\sum_{i=0}^m \binom{m}{i} (1-p)^{m-i} p^i \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \right] \\ &= \left[\sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \right] + \left[\sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m}{i} (1-p)^{m-i} p^i \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \right] \\ &\quad + \left[\sum_{i=\lfloor m/2 \rfloor + 1}^m \binom{m}{i} (1-p)^{m-i} p^i \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \right]. \end{aligned}$$

The proof proceeds by handling the three parts on the right-hand side of the inequality above separately.

For the first part, using lemma 2, we can get

$$\begin{aligned} \mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{U}) &= \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \mathbb{E}_{S_1 \sim \mathbf{D}^i} \left[\mathbb{E}_{S_2 \sim \mathbf{U}^{m-i}} [\hat{\mathfrak{R}}_{S_1 \cup S_2}(\mathcal{F})] \right] \\ &\geq \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \mathbb{E}_{S_1 \sim \mathbf{D}^i} \left[\mathbb{E}_{S_2 \sim \mathbf{U}^{m-i}} \left[\frac{i}{m} \hat{\mathfrak{R}}_{S_1}(\mathcal{F}) \right] \right] \\ &= \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \mathbb{E}_{S_1 \sim \mathbf{D}^i} \left[\frac{i}{m} \hat{\mathfrak{R}}_{S_1}(\mathcal{F}) \right] \\ &= \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}). \end{aligned}$$

The second part can also proceed using lemma 2. It is essentially upper-bounded by the first part. By the fact that $i \leq m-i$ for $0 \leq i \leq \lfloor m/2 \rfloor$, we obtain

$$\begin{aligned} \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m}{i} (1-p)^{m-i} p^i \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) &\leq \sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m}{i} (1-p)^{m-i} p^i \frac{m-i}{m} \mathfrak{R}_{m-i}(\mathcal{F}, \mathbf{D}) \\ &= \sum_{i=m-\lfloor m/2 \rfloor}^m \binom{m}{i} (1-p)^i p^{m-i} \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \\ &\leq \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \\ &\leq \mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{U}) \end{aligned}$$

The third part takes advantage of the assumption that $p \leq 0.5$. We know that for $\lfloor m/2 \rfloor + 1 \leq i \leq m$, the assumption $p \leq 0.5$ implies

$$(1-p)^{m-i} p^i \leq \frac{p}{1-p} (1-p)^i p^{m-i}.$$

Therefore, using the first part, we achieve

$$\begin{aligned} \sum_{i=\lfloor m/2 \rfloor + 1}^m \binom{m}{i} (1-p)^{m-i} p^i \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) &\leq \sum_{i=\lfloor m/2 \rfloor + 1}^m \binom{m}{i} \frac{p}{1-p} (1-p)^i p^{m-i} \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \\ &= \frac{p}{1-p} \sum_{i=\lfloor m/2 \rfloor + 1}^m \binom{m}{i} (1-p)^i p^{m-i} \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \\ &\leq \frac{p}{1-p} \sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \\ &\leq \frac{p}{1-p} \mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{U}). \end{aligned}$$

By combining all the three parts above, we establish

$$\begin{aligned} \mathfrak{R}_m(\mathcal{F}, \mathbf{D}) &\leq \left[\sum_{i=0}^m \binom{m}{i} (1-p)^i p^{m-i} \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \right] + \left[\sum_{i=0}^{\lfloor m/2 \rfloor} \binom{m}{i} (1-p)^{m-i} p^i \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \right] \\ &\quad + \left[\sum_{i=\lfloor m/2 \rfloor + 1}^m \binom{m}{i} (1-p)^{m-i} p^i \frac{i}{m} \mathfrak{R}_i(\mathcal{F}, \mathbf{D}) \right] \\ &\leq \left(1 + 1 + \frac{p}{1-p} \right) \mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{U}) \\ &= \frac{2-p}{1-p} \mathfrak{R}_m(\mathcal{F}, (1-p)\mathbf{D} + p\mathbf{U}). \end{aligned}$$

The proof for theorem 3 is therefore concluded. \square

The relation between $\mathfrak{R}_n(\mathcal{F}, \mathbf{D})$ and $\mathfrak{R}_m(\mathcal{F}, \mathbf{D})$ is achieved by the following theorem.

Theorem 4 (Concentration inequality of subset Rademacher complexity). *Assume in solving the joint problem we obtained m independently and identically distributed samples. Let the random number n represent the number of supervised sample obtained among these m joint samples with a proportion probability of $1-p$. Then, with probability at least $1-\delta$, the following holds*

$$\mathfrak{R}_n(\mathcal{F}, \mathbf{D}) \leq \frac{\mathfrak{R}_m(\mathcal{F}, \mathbf{D})}{1-p - \sqrt{\frac{\log(1/\delta)}{2m}}}, \quad (24)$$

for large enough m such that

$$1-p - \sqrt{\frac{\log(1/\delta)}{2m}} > 0. \quad (25)$$

Proof. Using lemma 2, we only need to prove an upper bound for m/n . Since we know that n follows a binomial distribution with mean $(1-p)m$, using Hoeffding's inequality (Hoeffding (1963), Serfling (1974)), we can obtain

$$\Pr[n \leq (1-p-\epsilon)m] \leq \exp(-2\epsilon^2 m),$$

or put differently,

$$\Pr\left[\frac{m}{n} \leq \frac{1}{1-p-\epsilon}\right] \geq 1 - \exp(-2\epsilon^2 m).$$

The inequality is obtained by setting $\delta = \exp(-2\epsilon^2 m)$. The proof assumes that m is large enough such that

$$1-p - \sqrt{\frac{\log(1/\delta)}{2m}} > 0.$$

\square

As a result, theorem 2 can be obtained by directly combining theorem 3 and theorem 4.